Shape of Compactifications (Shape Theory and Topological Spaces)

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Shape of compactifications

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All of spaces are assumed to be metrizable and hence compactifications are metrizable ones.

General questions are as follows.

1) Has a good space always a good compactification ?

2) Does there exist a good space every compactification of which is bad ?

3) Does there exist a bad space having a good compactification ?

Here by "good" we mean "with good shape property".

Example 1. Let X be a countable discrete space. Then for every compactum Y there is a compactification αX such that the remainder $\alpha X - X \approx Y$ and Y is a retract of αX.

Example 2. There are compactifications $\alpha_i R$, i = 1,2,3, of the real line R:

1) $\alpha_1 R$ is an FAR and $\alpha_1 R - R$ consists of two segments.

2) $\alpha_2 R$ is an FANR and $\alpha_2 R - R$ consists of two circles.

3) $\alpha_3 R$ is not pointed 1-movable. To construct $\alpha_3 R$, let M be a solenoid and let f be a 1-1 map from R onto one composant of M. Put $X = \{(f(x), 1/|x|+1 : x \in R\} \subset M \times I$ and $\alpha_3 R = \mathrm{Cl}_{M\times I} X$. Then $X \approx R$ and $\alpha_3 R$ is a compactification of R. Since M is a retract of $\alpha_3 R$, $\alpha_3 R$ is not pointed 1-movable.

Let us use the following notations.

$R_+ = [0,\infty)$; $E^* = \overbrace{\sqrt{R_+}}^{\beta R_+ -}$, where βR_+ is the Stone-Čech compactification of R_+; $I^* = \beta(I\times R_+) - I\times R_+$. E^* is said to be a *Čech* 0-cell. I^* contains two Čech 0-cells $E_0^* = (\mathrm{Cl}_{\beta(I\times R_+)}\{0\}\times R_+)\cap I^*$ and $E_1^* = (\mathrm{Cl}_{\beta(I\times R_+)}\{1\}\times R_+)\cap I^*$.

A space X is said to be *Čech path connected* if for $x_0, x_1 \in X$ there is a map $f : I^* \to X$ (called a *Čech path*) such that $f(E_i^*) = x_i$, $i = 0,1$. X is said to be *locally Čech path connected* if for $x \in X$ and a neighborhood U of x there is a neighborhood $V \subset U$ of x such that any two points of V are connected by a Čech path in U. The following characterization of pointed 1-movability is given in [1].

Theorem 1. *The followings are equivalent for a continuum* X.

(1) X *is pointed* 1-*movable*.

(2) X *is Čech path connected*.

(3) *Every map* $f : E_0^* \cup E_1^* \to X$ *is extendable over* I^*.

As a consequence we have the following Krasinkiewicz's theorem.

Corollary (Krasinkiewicz [3]) *A continuous image of a pointed* 1-*movable continuum is pointed* 1-*movable*.

As an application of Theorem 1 we have the following theorem for compactifications of pointed 1-movable spaces.

Theorem 2 [2] *Let* X *be a connected, locally Čech path connected, locally compact space. If* αX *is a compactification of* X *such that each component of the remainder* $\alpha X - X$ *is pointed* 1-*movable, then* αX *is pointed* 1-*movable*.

Corollary 1. Let X be a space in Theorem 2. Then the Freudenthal compactification and the one point compactification of X are pointed 1-movable.

Corollary 2. Let X be a locally connected locally compact space. If αX is a compactification of X such that every component of the remainder $\alpha X - X$ is pointed 1-movable, then αX is pointed 1-movable.

In Theorem 2, Collaries 1 and 2, as shown by Example 2, we can not omit pointed 1-movability of the components of the remainder. Also, we can not replace pointed 1-movablity by pointed r-movability ($r \geqq 1$).

Example 3. Let $r > 0$. Let $\{S_i, f_{i,i+1} : i=1,2,\dots\}$ be the inverse sequence of r-sphere S_i with bonding map $f_{i,i+1}$ of fixed degree > 1. Let X' be the telescope associated with $\{S_i\}$. Put $X = C(S_1) \cup X'$, where $C(S_1)$ is the cone over S_1. Then the one point compactification (= the Freudenthal compactification) of X is not r-movable.

Problem 1. Does there exist a locally compact ANR X such that every compactification of X is not movable ?

Problem 2. Has every (pointed r-) movable locally compact space a (pointed r-) movable compactification ?

Problem 3. Has every locally compact metrizable ANSE a movable compactification ?

Problem 4. Has every locally compact metrizable ASE a compactification which is ASE ?

References

[1] Y.Kodama, On fine shape theory III.

[2] ————, Compactification of pointed 1-movable spaces.

[3] J.Krasinkiewicz, Continuous images of continua and 1-movability, Fund.Math., 98(1978), 141-164.